1 dim $\mathbb{R} \to \mathbb{R}$ ✓ (BV function ; 构造)

n dim 可导性: Rademacher $t \mapsto f(x+te)$ 对 $\mathcal{L}^1(B_e \cap \{x+te, t \in \mathbb{R}\}) = 0$

1. directional derivative (by Fubini) $\mathcal{L}^n(B_e) = 0$
2. differentiability ~ existence of sufficiently many partial der.

$(e_1, \cdots, e_n)$ 正交 $\nabla f = (\partial_1 f, \cdots, \partial_n f)$

充分性? pf $\forall e$. $\partial_e f(x) = e \cdot \nabla f(x)$ $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$

对 $f: \mathbb{R}^n \to \mathbb{R}^m$ is non-differentiable on E:

— sufficiently many partial derivatives ~ differentiability

Definition (Porous set)

Definition 1.2. A set $E \subset \mathbb{R}^n$ is *porous at a point* $x \in E$ if there is a $c > 0$ and there is a sequence $y_n \to 0$ such that the balls $B(x + y_n, c|y_n|)$ are disjoint from E. The set E is *porous* if it is porous at each of its points, and it is called *σ-porous* if it is a countable union of porous sets.

Theorem.

Theorem 1.3 ([3]). *Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then the set of those points at which f is not differentiable but it is differentiable in n linearly independent directions is σ-porous.*

Remark: ①. σ-porous set have Lebesgue measure zero

Lebesgue density theorem

②. a set E is porous at $x \in E$ $\Rightarrow$ dist(x, E) non-differentiable at x

Find all Lebesgue null set for which there is a non-differ Lip funct.

→ consider functions not having enough many directional derivatives.

History: (D. Preiss)

任意一个 Lip 函数在 E 上有一点可微.

Theorem 1.5 ([9]). *There is a Lebesgue null set $E \subset \mathbb{R}^2$ such that every Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable in at least one point of E.*

更确切.

▶ [Preiss, 1990] Any G_δ set E containing dense set of lines in $\mathbb{R}^n$ is a *universal differentiability set*, i.e. any Lipschitz $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at some points in E.

Theorem 1.6. *For every Lebesgue null set $E \subset \mathbb{R}^2$ there is a Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}^2$ which is not differentiable at any point $x \in E$.*

(Rademacher inverse)

Def. (uniformly purely unrectifiable) the set of points at which a lip function may be differlable in no direction.

Remark: ① form a σ-ideal.

② uniformly purely unrectifiable set are purely unrectifiable

— 先考虑 $f: \mathbb{R}^2 \to \mathbb{R}^m$

if f is not differentiable at the points of $E \subset \mathbb{R}^2$, then at each point $x \in E$ except for a uniformly purely unrectifiable set, there is a *unique* differentiability direction $\tau(x)$ of f. Moreover, this direction is determined by the geometry of the set E, it is independent of the function f: for any other Lipschitz function g, the direction constructed using f and g agree at each point of E except for a uniformly purely unrectifiable set. Indeed, if E is contained in the non-differentiability set of both $f: \mathbb{R}^2 \to \mathbb{R}^{m_1}$ and $g: \mathbb{R}^2 \to \mathbb{R}^{m_2}$, then the direction τ defined by the function $h = (f, g): \mathbb{R}^2 \to \mathbb{R}^{m_1 + m_2}$ must coincide with the directions defined by f or g, whenever f, g and h have a unique direction of differentiability.

(+ 1.6)

Corollary 1.7. *For every planar Lebesgue null set E, at each point $x \in E$ there is a direction $\tau(x)$ with the following property: every Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$ at every $x \in E$, except at a uniformly purely unrectifiable set of points. This direction is determined uniquely, except for a uniformly purely unrectifiable set.*

Remark. **4. Theorem 1.** *Given any measurable plane set E, $|E|<\infty$, we can construct a set L of lines such that*

(i) *through each point of E passes at least one line of L,*

(ii) $|L| = |E|$.

— $f: \mathbb{R}^n \to \mathbb{R}^m$

Notation: **Notation.** We denote by $\mathcal{N}_{n,k}$ the σ-ideal of subsets of $\mathbb{R}^n$ generated by sets for which there is a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ differentiable in at most k linearly independent directions.

$\Rightarrow$ $\mathcal{N}_{n,n-1}$ sets are k-purely unrectifiable

Def. tangent field.

Definition 1.8. $\tau: E \to G(n,k)$ is called a *k-dimensional tangent field* of a set E if every Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$ at all $x \in E$ except those belonging to an $\mathcal{N}_{n,k-1}$ set.

Theorem 1.9. *Every set $E \in \mathcal{N}_{n,k}$ has a k-dimensional tangent field. Moreover, the tangent field is unique up to an $\mathcal{N}_{n,k-1}$ set.* — Cor 1.7 plus

— tangent field equal def.

一来源:

Proposition 1.10. *The set of (directional) non-differentiability of a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ can be written as a countable union of sets E, for each of which we may find a direction u and numbers $a<b$ such that*

$$\liminf_{t\to 0} \frac{f(x+tu)-f(x)}{t} < a < b < \limsup_{t\to 0} \frac{f(x+tu)-f(x)}{t}.$$

Pf. $a, b \in \mathbb{Q}\times\mathbb{Q}$. $u \in S^{n-1}$ 分量有理

给定 x. 在 u 方向不可导. $\Rightarrow$ $f(x+tu) - Lt|u_n| \le f(x+t(u+u_n)) \le f(x+tu) + Lt|u_n|$

$\liminf \frac{f(x+t(u+u_n))-f(x)}{t} \le \liminf \frac{f(x+tu)-f(x)}{t} + L|u_n|$

$\limsup \frac{f(x+t(u+u_n))-f(x)}{t} \ge \limsup \frac{f(x+tu)-f(x)}{t} - L|u_n|$ $\square$

$\mathcal{L}^1$ 意义下.

$\Rightarrow$ f Lip. $\Rightarrow$ $E_{u,a,b}$ null on every line in direction u.

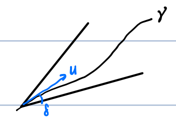
& every curve $\gamma: \mathbb{R} \to \mathbb{R}^n$ $\underbrace{|\gamma'-u|}$ small enough

$L|u_n| + a < b - L|u_n|$

$|u_n| < \frac{b-a}{2L}$

Do better: if $\delta > 0$ small enough, for every $\varepsilon > 0$ there is an open set $G \supset E$

s.t. the length of $G \cap \gamma$ is less than ε for every curve $\gamma: \mathbb{R} \to \mathbb{R}^n$ with $|\gamma' - u| < \delta$



Def (C-width): Given a convex cone C, the C-width of an $\underbrace{\text{open set}}$ G $= \sup_\gamma l(\gamma \cap G)$

Lip curve, $\gamma' \in C$ a.e. t

general: $\inf_{G \supset E \text{ open}} C\text{-width}(G)$

Def. **Definition 1.11.** If $E \subset \mathbb{R}^n$, we say that the mapping $\tau: E \to G(n,k)$ is a k-*dimensional tangent field* of E if for every cone C, the set of those points $x \in E$ for which $\tau(x) \cap C = \{0\}$ has C-width zero.

equal Def 1.8. :

$\Rightarrow$ the set where f is not differentiable can be covered by countably many sets, each of which has width zero w.r.t. some cone.

$\Rightarrow$ full description ?

Theorem 1.12. *For every set $E \subset \mathbb{R}^n$, the following are equivalent:*

(i) *There is a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that is non-differentiable at any point of E.*
(ii) *There is a sequence (possibly infinite) of Lipschitz functions $f_j\colon \mathbb{R}^n \to \mathbb{R}$ such that at every point of E at least one of the f_j is non-differentiable.*
(iii) *The set E is in $\mathcal{N}_{n,n-1}$.*
(iv) *The set E has an $(n-1)$-tangent field.*
(v) *If $n \le 2$: E has Lebesgue measure zero.*

every Lebesgue null set is in $\mathcal{N}_{n,n-1}$ for $n>2$?

$m<n$: there is a null set $E \subset \mathbb{R}^n$ s.t. every lip is differentiable at some point of E. ?

yes : $1=m<n$
$2=m<n$

How we can construct a non-differentiable function for a given (small) set E

1 dim :

Theorem 1.13 (Zahorski). *For any G_δ set $E \subset \mathbb{R}$ of Lebesgue measure zero there is a Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(f) \le 1$ which is differentiable at every point $x \notin E$ and*

$$\liminf_{t\to 0} \frac{f(x+t)-f(x)}{t} = -1 < 1 = \limsup_{t\to 0} \frac{f(x+t)-f(x)}{t}$$

for every $x \in E$.

还记得吗 : $E \subset \mathbb{R}$ is the set of points of non-differentiability of some Lip function $f: \mathbb{R} \to \mathbb{R}$

$\Leftrightarrow$ $m(E)=0$ $E \in G_{\delta\sigma}$ (a union of countably many G_δ sets

Construction. ($\forall$ open set is countable union of disjoint open intervals.) Given E : null.

$G_1 \supset G_2 \supset \cdots \supset E$ so small that G_k is small in every component of G_{k-1}

(若 E compact , $\{G_n\}$: $l(G_n) \le 2^{-n} l(I)$ $\forall I$: connected component of G_{n-1} (finite many)
not $l(G_n \cap I) \le 2^{-n} l(I)$)

$f_k(x) = m\big((-\infty, x) \cap G_k\big)$ $f^{(n)} := \sum_{k=1}^{n} (-1)^k f_k(x)$ is 1-Lip $\Rightarrow$ f Lip + 证明.

$f^{(\infty)} = \sum_{k=1}^{\infty} (-1)^k f_k(x)$, $\forall x \in E$. odd n : $\forall y \in I_n$ (closure of the connected component of A_n contain x)

$$\left|\frac{f(y)-f(x)}{y-x}\right| = \left|\sum_{k=n}^{\infty} \frac{(-1)^k[f_k(y) - f_k(x)]}{y-x}\right| \ge \left|\frac{f_n(y)-f_n(x)}{y-x}\right| - \sum_{k=n+1}^{\infty} \frac{|f_k(y)-f_k(x)|}{|y-x|}$$

$$= 1 - \sum_{k=n+1}^{\infty} \frac{l(A_k \cap [x,y])}{|y-x|}$$

$$\ge 1 - \sum_{k=n+1}^{\infty} \frac{l(A_k \cap I)}{|y-x|} \ge 1 - \sum_{k=n+1}^{\infty} \frac{l(I)\cdot 2^{-k}}{|y-x|} \ge 1 - \frac{l(I)\cdot 2^{-n}}{|y-x|}$$

取 y_n $|y_n - x| \ge \frac{l(I)}{4}$ $\Rightarrow$ $\frac{f(y_n)-f(x)}{y-x} \ge 1-2^{2-n}$

even n , $\left|\frac{f(y)-f(x)}{y-x}\right| = \left|\sum_{m=n}^{\infty} \frac{f_m(y)-f_m(x)}{|y-x|}\right| \le 2^{2-n}$

If $E \in G_\delta$, at points of $\mathbb{R} \setminus E$: upper and lower deri of f differ by no more than 2ε. not f is differentiable

higher dim. f_k : for an open set $G \subset \mathbb{R}^n$ of (small) C-width w and unit vector e from interior of C

construct a function w : $\mathbb{R}^n \to \mathbb{R}$ s.t. $\mathrm{Lip}(w)$ bounded (depending on C and e)

$\{$ $w(y) \ge w(x)$ if $y-x \in C$

$w(x+te) = w(x)+t$ segment $[x, x+te]$ lies in G

$0 \in w(x) \in w$ all $x \in \mathbb{R}^n$

$\Rightarrow$ w has directional deri 1 in the direction e at each $x \in G$

from global point of view, w looks like have deri zero.

(跳过 w 的定义)

Theorem 1.15. *For every $\tilde{\varepsilon} > 0$ and for every set E which is G_δ and uniformly purely unrectifiable there is a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $\mathrm{Lip}(f) = 1$;

(ii) *f is $\tilde{\varepsilon}$-differentiable on $\mathbb{R}^n \setminus E$, that is, for every $x \in \mathbb{R}^n \setminus E$ there is $r > 0$ and a vector u such that*

$$|f(x) - f(y) - \langle u, y - x\rangle| \le \tilde{\varepsilon}|y - x| \quad \text{for all } y \in B(x, r),$$

(iii) *for every $x \in E$, $\eta \in B(0,1) \subset \mathbb{R}^n$ and $\varepsilon > 0$ there is an $r < \varepsilon$ such that*

$$|f(y) - f(x) - \langle \eta, y - x\rangle| \le \varepsilon r \quad \text{for all } y \in B(x, r).$$

In particular, f is not differentiable at the points of E, it is not even ε-differentiable for any $\varepsilon < 1$.

conjecture: $N_{n,0} = G_{\delta\sigma}$?

the sets of points of k-dim differ can be characterised as:

Theorem 1.16. (i) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function, and for each $x \in \mathbb{R}^n$ choose $\tau(x)$ to be a maximal dimensional subspace such that the restriction of f to $x+\tau(x)$ is differentiable at x. For each $0 \le k \le n-1$, let E_k denote the set of those points at which $\dim \tau(x) = k$. Then $E_k \in \mathcal{N}_{n,k}$.*

(ii) *Let $E_k \subset \mathbb{R}^n$ be an $\mathcal{N}_{n,k}$ set for some $0 \le k \le n-1$. Then there is a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}^{k+1}$ and a k-tangent field τ of E_k such that f is not differentiable at any $x \in E_k$ in any direction e that is orthogonal to $\tau(x)$.*

Analogy of Thm 1.15

Theorem 1.17. *For each $0 \le k < n$ there is a constant $c_{n,k} > 0$ such that, whenever $l > k$, $\varepsilon > 0$ and E is a G_δ, $\mathcal{N}_{n,k}$ subset of $\mathbb{R}^n$, then there is a function $f\colon \mathbb{R}^n \to \mathbb{R}^l$ with $\mathrm{Lip}(f) \le 1$ which is ε-directionally differentiable at every point of $\mathbb{R}^n \setminus E$ and has the property that for every $x \in E$ there are k-dimensional linear subspaces V, W of $\mathbb{R}^n, \mathbb{R}^l$, respectively, so that for any unit vectors $v \in V^\perp$ and $w \in W^\perp$,*

$$\limsup_{t \searrow 0} \frac{\langle f(x+tv) - f(x), w\rangle}{t} - \liminf_{t \searrow 0} \frac{\langle f(x+tv) - f(x), w\rangle}{t} \ge c_{n,k}\,.$$