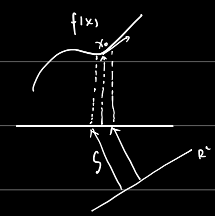
Background.

— k-dimensional surface area in R^n.

define its density in every k-dim linear subspace.



$A : G(n,k) \to R_+ \quad (\in C)$. — associated surface area formula $\underbrace{\mathrm{Area}_A}_{\text{}}(S) = \int_S A(T_x S)\, dm(x)$

regarded as points in $G(n,k)$

k-dim Euclidean surface area.

Smooth (or Lip) surface.

— especially. Busemann-Hausdorff definition of area:

$A^{bh}(P) = \dfrac{\varepsilon_k}{m_k(P \cap B)}$ ↳ $B \subset R^n$ is the unit ball of the norm $\|\cdot\|$

↓ k-dim Euclidean area.

⇒ geometric meaning: for embedded surfaces:

· k-dim Hausdorff measure of the surface

⇒ the area of the norm's unit ball (in P) $= \varepsilon_k$.

$S : R^k \to R^n$

$Df(x_0)\, S : R^k \to T_{x_0}$

等距同构

$\det(S) = Jf(x_0)^{-1}$

$Jf(x_0) \cdot \det S = 1$.

⇒ $\mathcal{H}^n(S^{-1}(A)) = J_f(x_0)\, \mathcal{H}^n(A)$

$J_f = [\det(S)]^{-1}$

— Affine-invariant definition.

define (translation invariant) k-dimensional density in V:

$A : GC_k(V) \to R_+$ sym. positively homogeneous ($A(\lambda 6) = |\lambda|\, A(6)$)

Busemann-Hausdorff area in a normed space $(V, \|\cdot\|)$:

L: $e_1 \to v_1$, $e_2 \to v_2$. $\det(L) = \dfrac{\varepsilon_k}{m(L^{-1}(B))} = |v_1 \wedge v_2|$.

$A^{bh}(v_1 \wedge \cdots \wedge v_k) = \dfrac{\varepsilon_k}{m_k(L^{-1}(B))}$ (= k-dim Hausdorff measure of the parallelotope spanned by the vectors $v_1, \cdots, v_k$.)

↓ unit ball

$L : R^k \to V \quad L(e_i) = v_i \quad$ standard basis $(e_1, \cdots, e_k)$

Remark: $A^{bh}(S) = \mathrm{Area}_{A^{bh}}(S)$

— Lip chain $S = \sum a_i S_i \quad S_i : \Delta \to R^n \quad a_i : \mathbb{Z}, R, \mathbb{Z}_2$

define $\mathrm{Area}_A(S) = \sum |a_i|\, \mathrm{Area}_A(S_i)$ $\quad \begin{cases} |a_i| = 0 & \text{if } a_i = 0 \\ |a_i| = 1 & \text{otherwise.} \end{cases}$

· standard simplex.

$\{x \in R^k : x_0 + \cdots + x_{k-1} = 1,\ x_i \geq 0\}$

— convex of the density: if it can be extended to a convex function on $\Lambda^k V$

semi-elliptic of Area_A over R. $\mathbb{Z}$. $\mathbb{Z}_2$: whenever the boundary ∂S of a chain S over the respective ring

is equal to the boundary of a k-disc D embedded into an affine k-plane, one has $\mathrm{Area}_A(S) \geq \mathrm{Area}_A(D)$

convexity of A implies semi-ellipticity of Area_A over R and $\mathbb{Z}$.

— Busemann: co-dimension one: $\dim V = k+1$

— Today. $k=2$. supp. convex. In every finite-dimensional normed space V,

the two-dim Busemann-Hausdorff area density admits a convex extension to $\Lambda^2 V$.

⇒ implies the area is semi-elliptic over $\mathbb{Z}$. that is, planar discs minimize the area among orientable surfaces with the same boundary.

step 1. non-orientable area: the two-dimensional Busemann-Hausdorff area density is semi-elliptic over $\mathbb{Z}_2$.

⇔ every two-dim affine disc in V minimizes the Busemann-Hausdorff area among all compact Lip surfaces with the same boundary.

Def. V: finite-dim vector space.

$A: GC_k(V) \to \mathbb{R}_+$

$P \subset V$ k-dim linear subspace.

$\Rightarrow$ A calibrator for P with respect to A is a exterior k-form $\omega \in \Lambda^k V^*$ s.t. $\forall\, \sigma \in GC_k(V)$, one has $|\omega(\sigma)| \in A(\sigma)$

and if $\sigma \in \Lambda^k P$ inequality turns into equality.

① A admits a convex extension to $\Lambda^k V$ $\Leftrightarrow$ every 2-plane P admits a calibrator

— give explicit construction of such calibrators for $k=2$ $A = A^{bh}$

Fix P:

$\omega = \sum a_i H_{i_1} \wedge H_{i_2}$ （ω 的形式，$H_i \cdot H_j \in$ 线性映射。 $\sum a_i H_{i_1} \wedge H_{i_2}$

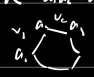
Fix $v_1, v_2 \in V$（且线性无关）。要有 $|\omega(v_1 \wedge v_2)| \in \underbrace{A^{bh}(v_1 \wedge v_2)}_{= \frac{\varepsilon_k}{m_k(L^{-1}(B))}}$ $\Leftrightarrow$ $|I^* \omega| \in \frac{\pi}{A(K)}$

$K = I^{-1}(B)$

$\Downarrow$ unit ball $\overset{①}{\Rightarrow}$ B is a polyhedron. & B sym.

$I: \mathbb{R}^2 \to V$. $e_i \to v_i$

Lem. $K = a_1 a_2 \cdots a_{2n}$. sym. $v_i = \overrightarrow{a_i a_{i+1}}$: $A(K) = \sum_{1 \le i < j \le n} |v_i \wedge v_j| = 1$ 1.



Lem'： $J: (\mathbb{R}^2)^* \to \mathbb{R}^2$ $|f_i \wedge f_j| = |J(f_i) \wedge J(f_j)|$. 且 $\sigma^{-1}(v_i) = S_{v_i} \underline{f_i}$.

在 v_i 上为 1.

$\Rightarrow$ $|\omega \circ I| = |\sum a_i f_{i_1} \wedge f_{i_2}|$

此外：F_i 在 B 所对应也为 1.

$\Rightarrow$ $\sum a_i \sigma(v_i) \wedge \sigma(v_j)$ 方向相同. 若 a_i 同向.

$\Rightarrow$ $|\sum a_i \sigma(v_i) \wedge \sigma(v_j)| = \sum a_i v_i \wedge v_j$

$a_i = \frac{1}{A(K)}$ 时. v.

$\Rightarrow$ $p_i = \frac{S_{v_i}}{A(K)} = \frac{A(\triangle o a_i a_{i+1})}{A(B \cap P_{v_1 v_2})}$ 所以. 若 v_1, v_2 在 P 上. 取等.

$$\Rightarrow \omega = \sum \frac{A(\triangle o a_i a_{i+1})}{A(B \cap P)} \text{———} F_i \wedge F_j.$$

$\Rightarrow$ 取 $\not=$. v. 考虑. $\underline{\sum p_i p_j f_i \wedge f_j}$ $\forall$ $\sum p_j = 1$.

$= \frac{1}{A(K)} \sum \lambda_i \lambda_j q_i q_j f_i \wedge f_j$

$= \frac{1}{A(K)} \sum \lambda_i \lambda_j |v_i \wedge v_j|$ K': $\lambda_i v_i$

$= \frac{A(K')}{A(K)}$

$1 = \sum p_i = \sum \lambda_i q_i = \underline{\frac{1}{A(K)} \sum \lambda_i} A(\triangle o a_i a_{i+1}) = \frac{1}{A(K)} V(K, K')$