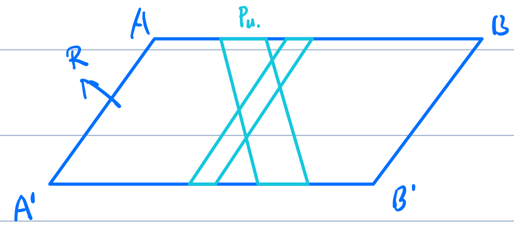
Q. Given any measurable plane set E, $|E| < \infty$, we can construct a set L of lines such that

i). through each point of E passes at least one line of L.

ii). $|L| = |E|$

Lem 6. Let $R = \text{▱} ABB'A'$ and K any closed [measurable] set contained in R. Then given positive numbers ε, ε_1, ε_2 we can construct a finite [countable] set of ▱ $P_u \subset R$ $(u = 1, 2, \dots)$ with two sides in AB and $A'B'$ s.t.



i). $K \subset \Sigma P_u$

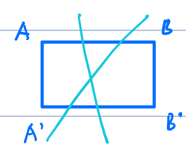
ii) $|\Sigma P_u - K| < \varepsilon$

iii) $|\underline{\lambda . \Sigma P_u}| < \varepsilon_2$ if λ is any line whose distance from $K \geq \varepsilon_1$

→ closed → ▱.

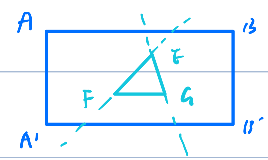
$\{$ $|\Sigma P_u - K| < \varepsilon$ →

Def. $R = \square ABB'A'$

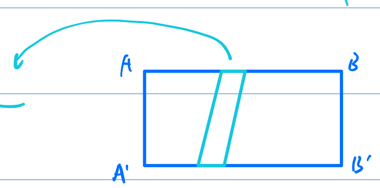


① A line meeting both AB and $A'B'$ is called admissible.

② A $\triangle EFG \subset R$ is called admissible if $\begin{cases} FG \parallel A'B' \\ \text{the sides } EF,\ EG \text{ produced each meet } AB,\ A'B' \text{ in interior points} \end{cases}$



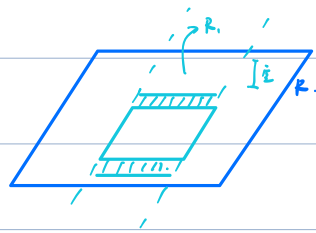
③ a ▱ having two sides in AB and $A'B'$ is called admissible.



④ a finite set of admissible ▱ is called an admissible set.

(Lem 5 (ii) can be omitted)

Lemma 4. Let $R = \square ABB'A'$, $R_1 = \square \in R$ with parallel sides. Then given $\varepsilon_1, \varepsilon_2 > 0$, we can cover R_1 with admissible set ΣP_u s.t. $|\lambda . \Sigma P_u| < \varepsilon_2$ $\forall \lambda$: $dist(\lambda, R_1) \geq \varepsilon_1$



Pf. ① 缩小 $\square$ 范围 ② 水平线去截.

$\Rightarrow$ **Lemma 5.** $\forall \varepsilon > 0$. + $|\Sigma P_u - R_1| < \varepsilon$ 面积

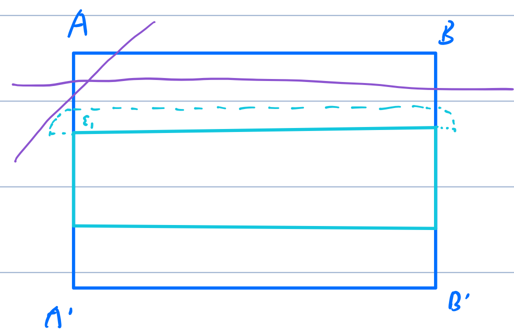
($\square \to \square$; half of the space)

Lemma 3. Given a $\square R_1 \subset \square R$ with two sides in AA' and BB', $\varepsilon_1, \varepsilon_2 > 0$. We can cover R_1 with an admissible set ΣP_u s.t. $|\lambda . \Sigma P_u| < \varepsilon_2$ $\forall \lambda$: $dist(\lambda, R_1) \geq \varepsilon_1$
meet R above R_1

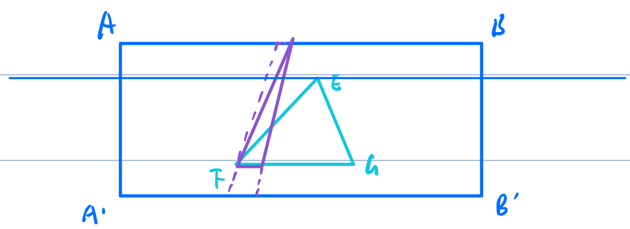
Lem 3 → 4. ① $\square \Leftrightarrow \square$.

② apply 3 to 4. ΣP_u

③ $\forall P_u$. apply 3 below

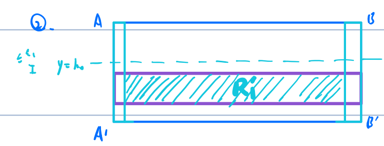


Lem 2. ad. $\Delta_0 \Rightarrow \{P_u\}$ admissible set i). $\Delta_0 \subset \Sigma P_u$. (ii) $|\lambda . \Sigma P_u| < \eta$. $\forall \lambda$: $\geq 2\theta$.
does not meet R below the horizontal through E.



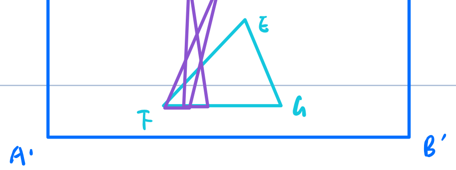
Lem 2 → Lem 3: ① "λ" in Lem 3 & ad. line inf —
cpt. ⟷ cpt.

②.



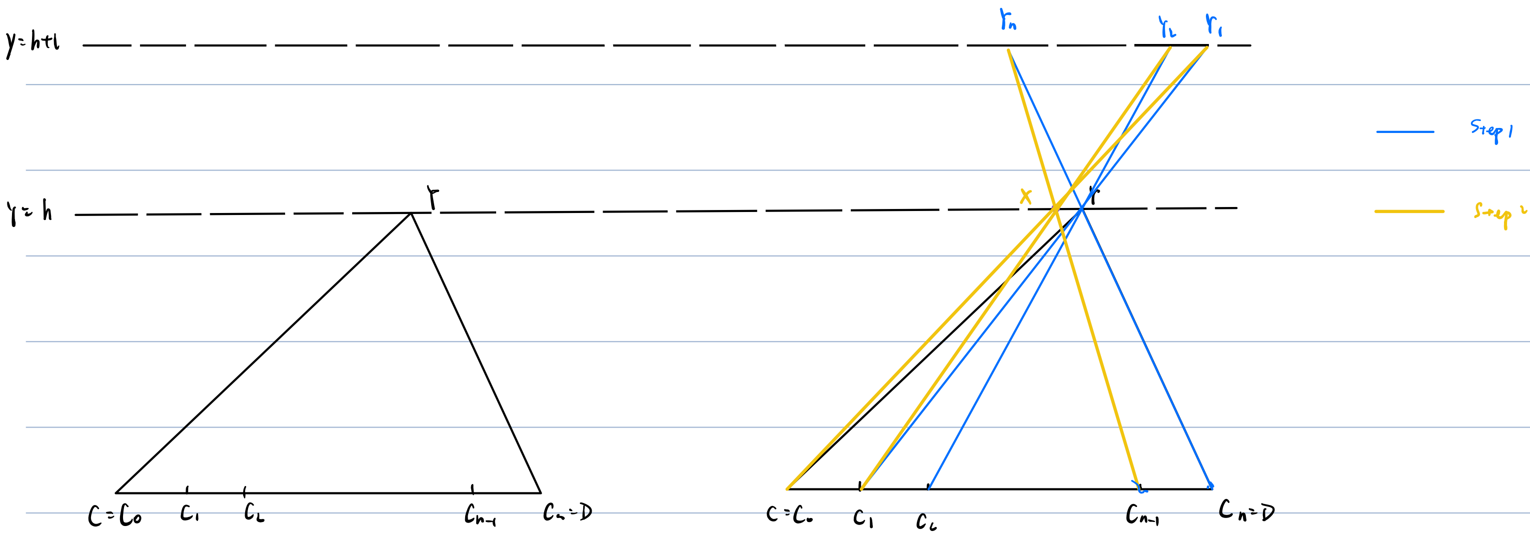
Lem 1. Given admissible $\Delta_0 = EFG$. $\eta, \theta > 0$. We can construct a finite set of admissible triangles whose bases are on FG and vertices on AB, s.t. (i). $\Delta_0 \in D$.

(ii). $|\lambda . D| < \eta$ $\forall \lambda$: makes an angle $\geq 2\theta$ with every admissible line it meets in R

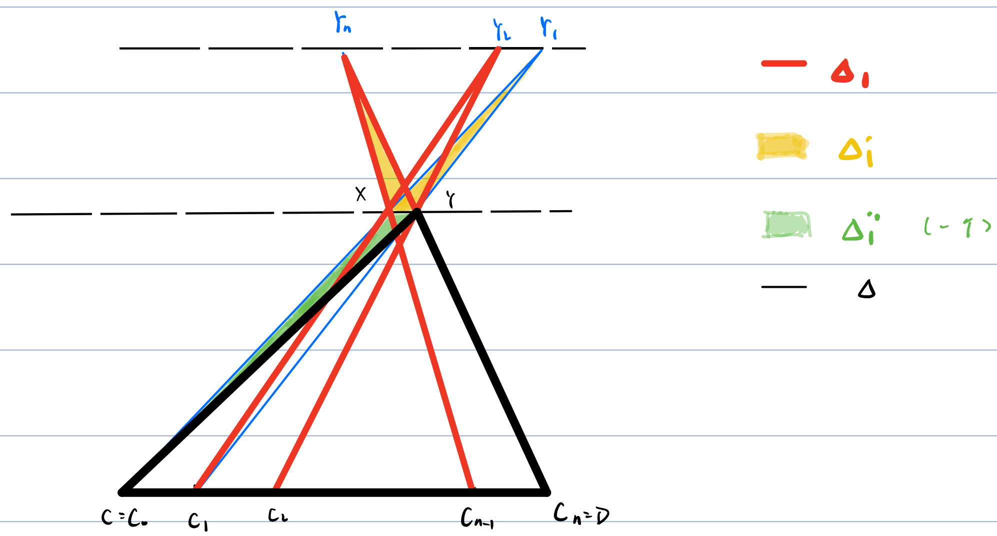


$\}$ $\lambda \cap \Delta_0 = \emptyset$

proof of Lem 1: "λ" satisfy Lem 1 "beautiful" Δ ad.



denote $\Delta_1 = \{ Y_v C_{v-1} C_v \ , v=1,\dots,n \}$ $\Delta_i' = \{ Y_v X Y \ , v=\dots \}$ $\Delta_i'' = XYC$



$\Rightarrow \begin{cases} \Delta < \Delta_1 \\ \Delta_1 - \Delta = \Delta_i' + \Delta_i'' \end{cases}$; all ad.

$\boxed{\text{Operation } O(\delta, l, \kappa):}$ $\Delta \to \Delta_1$: n so large that $|XY| < \kappa$

$\{ \ |\lambda \cdot \Delta_i''| < \delta$

— if Δ represents a finite set of ad. triangles whose base are on the same line and whose vertices are on the line $y = h$, Then $O(\delta, l, k)$ applied to Δ means $\Delta_1 = \bigcup \Delta_1^{(i)}$ $\quad \Delta_1^{(i)} = O(\frac{\delta}{m}, l, k)[\delta^i]$ $\quad \delta^i \in \Delta$ } ...

we have
$$\begin{cases} \Delta \subset \Delta_1 \\ \Delta_1 - \Delta \subset \Delta_1' + \Delta_1'' \\ |XY| < k \quad \text{if } |XY| \text{ is the upper side of a triangle of } \Delta_1'' \\ |\lambda . \Delta_1''| < m\delta' = \delta \end{cases}$$

pf: $\Delta_1 - \Delta \subset \Sigma(\quad) =$ ___

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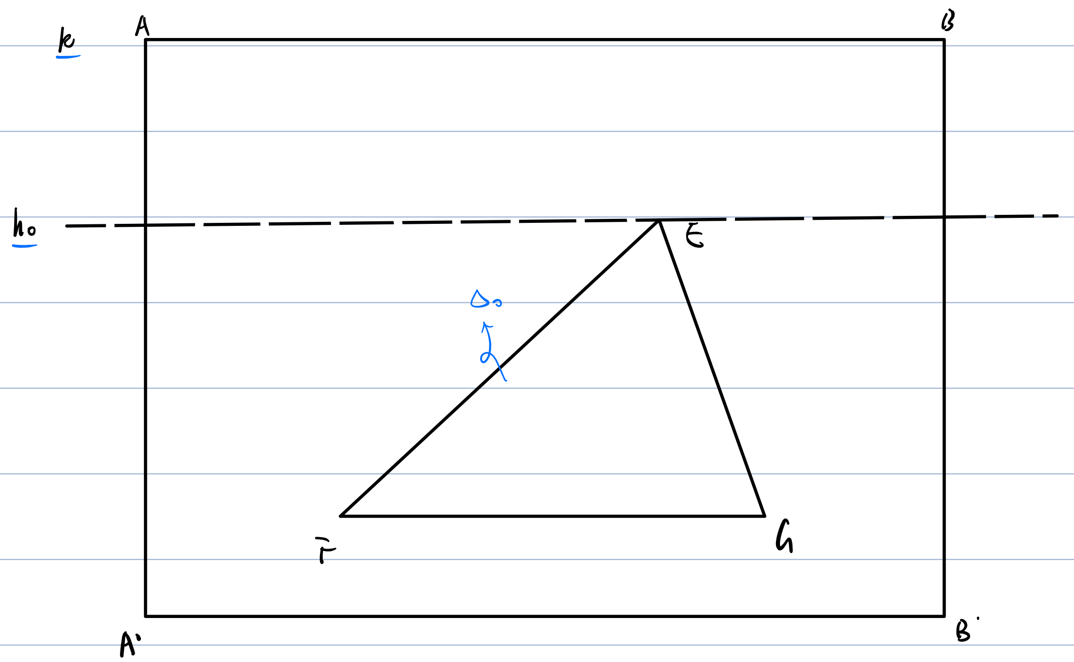
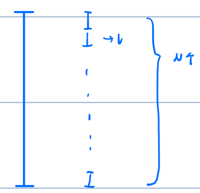
return to $\Delta_0 = EFG$





N so large that.

$$2l \frac{|FG|}{h_0} \sin 2\theta < \frac{1}{2}\eta$$
$$\begin{cases} \delta = \frac{2\eta}{2N} \\ k = 2l \sin\theta \end{cases}$$

We shall show $D = \Delta_N$ satisfies the condition of Lemma 1:

$$\begin{cases} \Delta_n \subset \Delta_{n+1} \\ \Delta_{n+1} - \Delta_n \subset \Delta_{n+1}' + \Delta_{n+1}'' \\ |XY| < 2l\sin\theta \quad \forall XY \text{ which is the upper side of a triangle of } \Delta_{n+1}'' \\ |\lambda . \Delta_{n+1}''| < \frac{\eta}{N} \quad \forall \lambda. \end{cases}$$

① $\Delta_0 \subset \Delta_N$

② $|\lambda . \Delta_N| < \eta \quad \forall \lambda$:

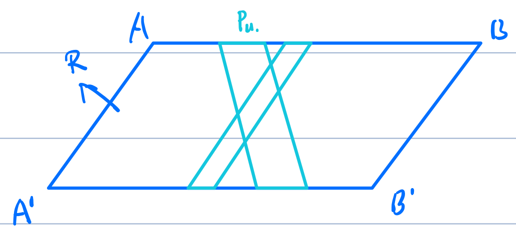
$$\Delta_N - \Delta_0 = (\Delta_1 - \Delta_0) + \cdots + (\Delta_N - \Delta_{N-1}) \subset (\Delta_1' + \cdots + \Delta_N') + (\Delta_1'' + \cdots + \Delta_N'')$$

$\Rightarrow \quad \lambda.\Delta_N = \lambda.(\Delta_N - \Delta_0) \subset \underbrace{\lambda.(\Delta_1' + \cdots + \Delta_N')}_{\text{(除为交集)}} + \underbrace{\lambda.(\Delta_1'' + \cdots + \Delta_N'')}_{\in \frac{1}{2}\eta.}$

Part II.

Review.

Lem 6. Let $R = \square ABB'A'$ and K any closed [measurable] set contained in R. Then given positive numbers $\varepsilon, \varepsilon_1, \varepsilon_2$ we can construct a finite [countable] set of $\square P_u \subset R$ $(u = 1, 2, \cdots)$ with two sides in AB and $A'B'$ s.t.



i). $K \subset \Sigma P_u$

ii) $|\Sigma P_u - K| < \varepsilon$

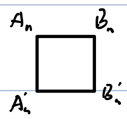
iii) $|\lambda.\Sigma P_u| < \varepsilon_2$ if λ is any line whose distance from $K \geq \varepsilon_1$

Thm 1. Given any measurable plane set E, $|E| < \infty$, we can construct a set L of lines s.t.

i) through each point of E passes at least one line of L

ii) $|L| = |E|$

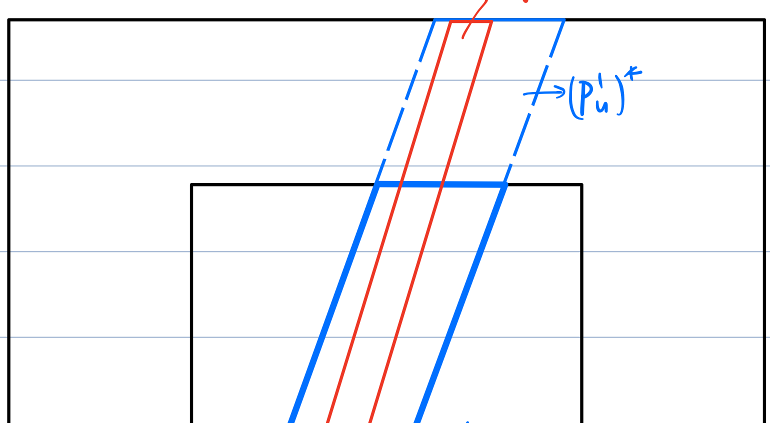
Pf. $S_n = \square A_n B_n B_n' A_n'$

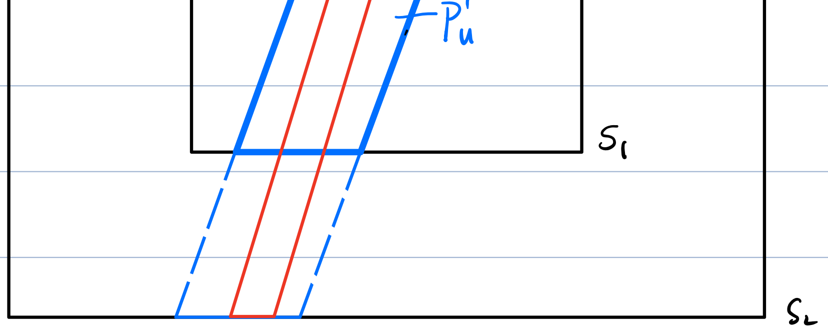


Suppose $E \subset S_1^{\circ}$

$\underline{\text{Lemma 6}}$ $\{P_u'\}$ bases in A_1B_1 and $A_1'B_1'$ s.t. $E \subset \sum_{u=1}^{\infty} P_u'$ $|\Sigma P_u' - E| < 1$

P_{uv}' $(v = 1, 2, \cdots)$ $\Rightarrow$ $\left|\sum_{v=1}^{\infty} P_{uv}' - E.P_u'\right| < 2^{-1}2^{-u}$





remember $\{P^1_{uv}\}$ $(u=1,2,\dots,\ v=1,2,\dots)$ as $\{P^2_w\}$ $(w=1,2,\dots)$ We clearly have :

(i) $E \subset \sum_{w=1}^{\infty} P^2_w$

(ii) $\left| \sum_{w=1}^{\infty} P^2_w - E \right| < \sum_{u=1}^{\infty} 2^{-1} 2^{-u} = 2^{-1}$ $\quad \left(\Rightarrow \left| \sum_{w=1}^{\infty} P^2_w \right| < |E| + 2^{-1} \right)$

(iii) $\forall p \in E . P^1_u$

$\Rightarrow \exists P^2_w : p \in P^2_w \ \& \ P^2_w \subset (P^1_u)^+$

Repeated application of this process

$\forall n$: $\{P^n_w\}$ $(w=1,2,\dots)$ of normal parallelograms with bases in A_nB_n and $A_n'B_n'$ s.t.

(i) $E \subset \sum_{w=1}^{\infty} P^n_w$

(ii) $\left| \sum_{w=1}^{\infty} P^n_w \right| < |E| + n^{-1}$

(iii) $p \in E . P^n_u$: $\exists P^{n+1}_w$ s.t. $p \in P^{n+1}_w$ $\quad P^{n+1}_w \subset (P^n_u)^+$

Denote $\mathcal{L}^n_w$ = the closed set of points covered by lines which meet both bases of P^n_w.

$\Rightarrow \mathcal{L}^n_w . S_n = P^n_w$; $\forall n > 0$:

(i) $E \subset \sum_{w=1}^{\infty} \mathcal{L}^n_w$

(ii) $\left| S_n . \sum_{w=1}^{\infty} \mathcal{L}^n_w \right| < |E| + n^{-1}$

(iii) $\forall p \in E . \mathcal{L}^n_u$: $\exists \mathcal{L}^{n+1}_w \subset \mathcal{L}^n_u$: $p \in \mathcal{L}^{n+1}_w$

$\to \bigcap \left(\sum_{w=1}^{\infty} \mathcal{L}^n_w \right)$

Denote $M = \prod_{n=1}^{\infty} \sum_{w=1}^{\infty} \mathcal{L}^n_w$

① $|S_n . M| < |E| + n^{-1}$ $\quad \forall n$

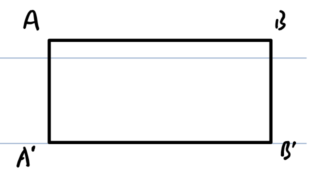
$\Rightarrow |M| \le |E|$

② $\forall p \in E$: choose a decreasing sequence of sets $\{\mathcal{L}^n_{w_n}\}$ each contain p

$\Rightarrow \prod_{n=1} L^{n}_{u_n}$ is a line through p.

Denote $\mathcal{L}$ the set of lines lying entirely in M. $E \subset \mathcal{L} \subset M$ ✓

A complicated version of Lem 6:



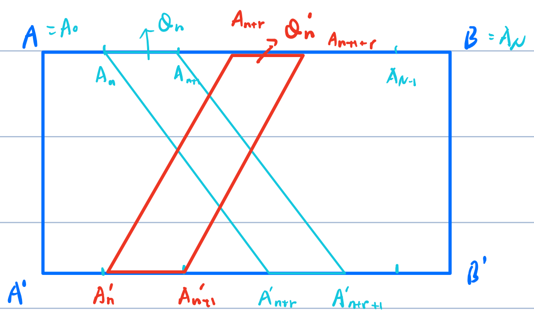
Lem 7. In Lem 6, we can impose the additional condition on the ▱:

① $\forall p \in K$, there exist two ▱ P_{u_1}, P_{u_2} each contained p in its interior, whose sides in AB, $A'B'$ are disjoint (call such ▱s 'separated')

② given $\alpha > 0$, we can make each ▱ P_u have its diagonals at angle $\le \alpha$ with AA'

Pf. Only need to prove this for $K = ▱ \subset R^\circ$ with sides parallel to those of R.

$2 \le r \le N-3$



$Q_n = A_n A_{n+1} A'_{n+r+1} A'_{n+r}$ $n = 1, 2, \cdots, N-r-2$

$Q'_n = A_{n+r} A_{n+r+1} A'_{n+1} A'_n$

suitable N, r:

(i) $\forall p \in K$ $\exists Q_n, Q'_m$ ① $p \in Q_n \cap Q'_m$

② separated.

(ii) diagonals of all ▱ make angle less than α with AA'

Thus, there are $2(N-r-2)$ Q_n, Q'_n. Apply Lem 6 $\Rightarrow$ ✓

Thm 2. $\forall$ measurable set E, $|E| < \infty$. $\forall \alpha > 0$, we can construct a set $\mathcal{L}$ of lines s.t.

(i) through $\forall p \in E$ pass $2^{\aleph_0}$ lines of $\mathcal{L}$ whose angles with the vertical are less than α.

(ii) $|\mathcal{L}| = |E|$

pf. We may suppose $E \subset S_1^\circ$.

Applying Lem 7 $\Rightarrow$ $\{P'_n\}$ = ▱ with bases in A_1B_1, $A'_1B'_1$ s.t.

(i). $E \subset P'_n$, $\forall p \in E$ is a interior point of two separated ▱

(ii) $|\sum P'_n - E| < 2^{-1}$

(iii) $\forall P'_n$ $\forall$ line $\subset P'_n$ make angle less than α with the vertical

...

$\Rightarrow P_w^n$: (i) $E \subset \Sigma P_w^n$

(ii) $|\Sigma P_w^n| < |E| + n^{-1}$

(iii) $\forall p \in E \cap P_w^n$: there are two separeted $\square \in \{P_w^{n+1}\}$:

① contained p in its interior

② lying in the extension of P_w^n

(iv) $\forall$ line $\subset P_w^n$ makes angle less the α with the vertical.

$\Rightarrow L_w^n$:

— call two sets $L_{w_1}^n$, $L_{w_2}^n$ separated if the corresponding $\square P_{w_1}^n$, $\square P_{w_2}^n$ separated.

① no line is contained both in $L_{w_1}^n$, $L_{w_2}^n$

② $\forall p \in E.(L_w^n)^\circ$, $\exists$ separated sets $L_{w_1}^{n+1}, L_{w_2}^{n+1} \subset L_w^n$ and $p \in L_{w_1}^{n+1}, L_{w_2}^{n+1}$

$\Rightarrow \forall p$: $L_{w_1}^1$. $L_{w_2}^1$ $\Rightarrow p \in (L_{w_1}^1)^\circ . (L_{w_2}^1)^\circ$

Rename $L_{w_1}^1 = L_1$, $L_{w_2}^1 = L_2$

L_{11} L_{12}

....

Corollary 1.7. *For every planar Lebesgue null set E, at each point $x \in E$ there is a direction $\tau(x)$ with the following property: every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable in the direction $\tau(x)$ at every $x \in E$, except at a uniformly purely unrectifiable set of points. This direction is determined uniquely, except for a uniformly purely unrectifiable set.*

Remark. There are null sets which are very far from being purely unrectifiable. For instance, R. O. Davies showed in [4] that every Borel set $B \subset \mathbb{R}^2$ can be covered by infinite straight lines without increasing its Lebesgue measure. One can even put continuum many lines through each of the points of B so that the union of these lines has the same measure as B. Now if $B = B_0$ is, say, a point, applying Davies's theorem iteratively, we can find $B_0 \subset B_1 \subset B_2 \subset \dots$ such that each B_k has continuum many lines through the points of B_{k-1}, and the sets B_k are Lebesgue null. Then $\bigcup B_k$ is also Lebesgue null, and it has continuum many lines through each of its points. What could be τ on $\bigcup B_k$? Since Lipschitz functions are differentiable along lines, at each line of the construction, τ must agree with the direction of the line at a.e. of its points. But there are continuum many lines at each point, how can we choose only one of these, so that along any given line at a.e. point we choose the direction of the given line and not one of the others?

Now, consider Lipschitz functions on $\mathbb{R}^n$.